Current fluctuations near to the 2D superconductor-insulator quantum critical point

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(Dated: February 5, 2008)

Systems near to quantum critical points show universal scaling in their response functions. We consider whether this scaling is reflected in their fluctuations; namely in current-noise. Naive scaling predicts low-temperature Johnson noise crossing over to noise power $\propto E^{z/(z+1)}$ at strong electric fields. We study this crossover in the metallic state at the 2d z=1 superconductor/insulator quantum critical point. Using a Boltzmann-Langevin approach within a $1/N$-expansion, we show that the current noise obeys a scaling form $S_j = T\Phi[T/T_{eff}(E)]$ with $T_{eff} \propto \sqrt{E}$. We recover Johnson noise in thermal equilibrium and $S_j \propto \sqrt{E}$ at strong electric fields. The suppression from free carrier shot noise is due to strong correlations at the critical point. We discuss its interpretation in terms of a diverging carrier charge $\propto 1/\sqrt{E}$ or as out-of-equilibrium Johnson noise with effective temperature $\propto \sqrt{E}$.

PACS numbers:

The notion of quantum criticality provides one of the few general approaches to the study of strongly correlated quantum many-body systems[1]. The scale invariance that characterizes the zero temperature critical point leads to characteristic, universal power-law dependences for various quantities in its proximity; these dependences can be computed within a continuum field theory. A number of theoretical works have calculated the impact of such universality upon conductivity[2, 3, 4]. These calculations provide robust experimental predictions. Two recent works of Dalidovch and Philips[5], and Green and Sondhi[6], have extended the analysis to see whether universality persists out of equilibrium. Surprisingly it does; at least in transport.

In this work we consider another experimentally measurable quantity: current fluctuations. Measurements of current fluctuations are usually restricted to mesoscopic samples in order that the relative fluctuations of current be significant. Mesoscopic samples have relatively few conducting channels giving a total conductance of order the conductance quantum e^2/h. Quantum-critical systems may also satisfy this criterion. For example, at the two-dimensional superconductor-insulator (SI)transition the conductance is also of order the conductance quantum[2, 3, 5, 6]. It would be revealing, therefore, to compare the current fluctuations at the metallic critical point of bosons with the more familiar results for non-equilibrium current noise in diffusive Fermi liquids[7]. Transport properties are readily measurable and theoretical predictions regarding their scaling and universality at two-dimensional quantum critical points (in charged systems such as SI, and Quantum Hall Effect transitions) have stimulated many experiments. Non-equilibrium current fluctuations may provide an independent window upon these quantum critical points.

There are two contributions to current noise; Johnson noise and Shot noise. Since it is a reflection of the fluctuation-dissipation relation, Johnson noise is expected to be unaffected by correlations between carriers. On the other hand, shot noise reflects the fact that charge is carried in quanta. If there are strong correlations between carriers, we expect it to be significantly altered.

The situation in a non-equilibrium steady state is trickier; Johnson and shot noise cannot easily be separated. On the one hand, one may identify an effective temperature scale T_{eff} for the out-of-equilibrium distribution and treat the current fluctuations as Johnson-like. Dimensional analysis suggests $T_{eff} \propto E^{z/(1+z)}$ and current noise proportional to $E^{z/(1+z)}$. On the other hand, one may identify the out-of-equilibrium noise with shot noise and a carrier charge that diverges as $E^{-1/(1+z)}$.

We present a microscopic calculation that recovers these results for the superfluid to insulator transition of two-dimensional bosons with particle-hole symmetry. We use a Boltzmann-Langevin[9, 10] approach within a $1/N$-expansion[4] to analyse current noise near this transition. This "kinetic theory of fluctuations" approach is known to give the correct result for the leading noise correlations in the diffusive metal, but needs to be modified for higher moments [11]. The system is considered to be in thermal contact with a substrate. This, in conjunction with the internal scattering in the system, leads to a uniform, non-equilibrium, steady-state distribution, provided that the length of the sample is greater than the correlation length[6]. This situation is quite different from that considered elsewhere, where the only coupling to the outside world is through the leads. It permits the current-noise properties to be entirely determined by the internal scattering and so to be universal.

Our main results are as follows: in thermal equilibrium, we recover Johnson noise with noise power $S_j = 4\sigma T$ (as we must in order to satisfy the equilibrium fluc-

tuations dissipation relation). This crosses over at very large electric fields to $S_j \propto \sqrt{E}$. This is strongly suppressed from the usual shot noise result of $S_j \propto j \propto E$ due to the strong correlations at the quantum critical point. This large field result may be considered as either shot noise with a carrier charge which diverges as $1/\sqrt{E}$ or alternatively as a non-equilibrium Johnson noise with effective temperature $T_{eff} = \sqrt{\hbar ceE}\pi^2/4k$. Between these two limits, we expect the current noise to obey a universal scaling form $S_j = T\Phi[T_{eff}(E)/T]$.

We begin by outlining the field theoretical formulation of the symmetric, bosonic, superfluid to insulator transition. We then give Boltzmann and Boltzmann-Langevin equations for the system in thermal equilibrium. The latter is used to deduce the Johnson noise. Next, we turn to the Boltzmann and Boltzmann-Langevin equations for the zero-temperature system under a finite electric field. This is used to deduce the out-of-equilibrium current-fluctuations. Finally we turn to a discussion of the implications of these results.

Field Theory The critical region of the symmetric superfluid to Mott insulator transition phase diagram is described by a charged scalar field with a quartic interaction[1, 2]:

$$\mathcal{H} = \int d^d x \left[\Pi^\dagger \Pi + \nabla\phi^\dagger \nabla\phi + m^2 \phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2\right], \quad (1)$$

where ϕ is the complex scalar field and Π is its conjugate momentum. These satisfy the usual commutation relations $[\phi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y})$. It is convenient to choose the bare interaction λ to have its fixed point value $u^*\Lambda^{3-d}$ with momentum cutoff Λ, [1] although we will not need the precise, regularization-dependent, value. At the zero-temperature critical point, the renormalized mass is zero which corresponds to a particular choice m^* of the bare mass. The effects of applying an electric field, $\mathbf{E}$, are included by minimally coupling to a vector potential; $\nabla\phi \to D\phi = (\nabla + ie\mathbf{A}/\hbar)\,\phi$. We choose the gauge $\mathbf{A} = \mathbf{E}t$. This is equivalent to a contact-free measurement (for example by placing the system on a cylinder and inducing an EMF on its surface by uniformly increasing the flux through it). We believe that our results carry over unchanged to the case with good contacts at the ends.

The normal modes of this Hamiltonian (in the absence of interaction) are charge density fluctuations. These occur with positive and negative charges, corresponding to decrease or increase in charge density from the average. We determine the current fluctuations near to the critical point by considering a Boltzmann equation for the occupation of these modes. We use $a^\dagger/b^\dagger$ and a/b to represent the creation and annihilation of positively/negatively charged density fluctuations.

The Boltzmann equation in thermal equilibrium and within a $1/N$-expansion of the Hamiltonian Eq.(1) is given by[2, 8]

$$\begin{aligned} &\left[\partial_t + (e\mathbf{E}/\hbar).\partial_\mathbf{k}\right] f_\mathbf{k} \\ &= -\int d\mathbf{q} \left[\begin{array}{c} \gamma_{\mathbf{kq}} f_\mathbf{k}\left(1+f_\mathbf{q}\right) - \gamma_{\mathbf{qk}} f_\mathbf{q}\left(1+f_\mathbf{k}\right) \\ \tilde\gamma_{\mathbf{kq}} f_\mathbf{k} f_{-\mathbf{q}} - \tilde\gamma_{\mathbf{qk}}\left(1+f_{-\mathbf{q}}\right)\left(1+f_\mathbf{k}\right) \end{array}\right] (2) \end{aligned}$$

$\gamma_{\mathbf{k},\mathbf{q}}$ and $\tilde\gamma_{\mathbf{k},\mathbf{q}}$ describe the matrix elements for particle-particle and particle-anti-particle scattering, respectively[12] and

$$f(\mathbf{k}, t) = \int d\mathbf{q} \langle a^\dagger_{\mathbf{k}+\mathbf{q}/2}(t) a_{\mathbf{k}-\mathbf{q}/2}(t)\rangle$$

denotes the distribution function, with a similar distribution $\tilde f(\mathbf{k}, t)$ for the negatively charged modes. Particle-hole symmetry implies the relation $f(\mathbf{k}, t) = \tilde f(-\mathbf{k}, t)$. We restrict our explicit consideration to the positively charged channel; final expressions include the negatively charged channel through appropriate factors of two.

The Boltzmann-Langevin equation is an equation describing the stochastic evolution of fluctuations, $\delta f_\mathbf{q}(\mathbf{r}, t)$, in the distribution function about its equilibrium or steady-state. In the present case, it is given by

$$\begin{aligned} &\left[\partial_t + \mathbf{v}.\partial_\mathbf{r} + (e\mathbf{E}/\hbar).\partial_\mathbf{q}\right]\delta f_\mathbf{k} = \gamma_\mathbf{k}\left[\delta f_\mathbf{k} - \delta \bar f_\mathbf{k}\right] + \eta_\mathbf{q}, \\ &= -\int d\mathbf{q}\left[\gamma_{\mathbf{kq}}\frac{1+f_\mathbf{q}}{1+f_\mathbf{k}}\delta f_\mathbf{k} - \gamma_{\mathbf{qk}}\frac{1+f_\mathbf{k}}{1+f_\mathbf{q}}\delta f_\mathbf{q}\right] + \eta_\mathbf{q} \\ &\quad -\int d\mathbf{q}\left[\tilde\gamma_{\mathbf{kq}}\frac{f_{-\mathbf{q}}}{1+f_\mathbf{k}}\delta f_\mathbf{k} - \tilde\gamma_{\mathbf{qk}}\frac{1+f_\mathbf{k}}{f_{-\mathbf{q}}}\delta f_{-\mathbf{q}}\right] \qquad (3) \end{aligned}$$

where

$$\begin{aligned} \delta \bar f_\mathbf{k} &= \int d\mathbf{q} M_{\mathbf{kq}} \delta f_\mathbf{q} \\ M_{\mathbf{kq}} &= \frac{\gamma_{\mathbf{qk}}}{\gamma_\mathbf{k}}\frac{1+f_\mathbf{k}}{1+f_\mathbf{q}} + \frac{\tilde\gamma_{-\mathbf{qk}}}{\gamma_\mathbf{k}}\frac{1+f_\mathbf{k}}{f_\mathbf{q}} \\ \gamma_\mathbf{k} &= \int d\mathbf{q}\left(\gamma_{\mathbf{kq}}\frac{1+f_\mathbf{q}}{1+f_\mathbf{k}} + \tilde\gamma_{\mathbf{kq}}\frac{f_{-\mathbf{q}}}{1+f_\mathbf{k}}\right). \qquad (4) \end{aligned}$$

The position and time labels of $\delta f_\mathbf{k}(\mathbf{r}, t)$ and $\eta_\mathbf{q}(\mathbf{r}, t)$ have been suppressed for compactness and the detailed balance conditions $\gamma_{\mathbf{q},\mathbf{k}} f_\mathbf{q}(1 + f_\mathbf{k}) = \gamma_{\mathbf{k},\mathbf{q}} f_\mathbf{k}(1 + f_\mathbf{q})$ and $\tilde\gamma_{\mathbf{q},\mathbf{k}}(1 + f_{-\mathbf{q}})(1 + f_\mathbf{k}) = \tilde\gamma_{\mathbf{k},\mathbf{q}} f_\mathbf{k} f_{-\mathbf{q}}$ have been used in order to simplify the right-hand-side. The time and momentum derivatives on the left-hand-side of Eq. (3) follow directly from the Boltzmann equation (2). The first term on the right hand side is the linearised form of the scattering integral. The second is a stochastic term describing fluctuations in occupation number. It is determined by assuming that the scattering processes are independently Poisson distributed. Under this assumption, the quadratic correlations of $\eta_\mathbf{q}(\mathbf{r}, t)$ are given by[13]

$$\begin{aligned} &\langle \eta_\mathbf{q}(\mathbf{r}, t)\eta_{\mathbf{q}'}(\mathbf{r}', t')\rangle \\ &= 2\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\left[\begin{array}{c} (2\pi)^2\delta_{\mathbf{q},\mathbf{q}'}\int d\mathbf{k}\gamma_{\mathbf{qk}} f_\mathbf{q}(1+f_\mathbf{k}) \\ -\gamma_{\mathbf{qq}'} f_\mathbf{q}(1+f_{\mathbf{q}'}) \\ +(2\pi)^2\delta_{\mathbf{q},\mathbf{q}'}\int d\mathbf{k}\tilde\gamma_{\mathbf{qk}} f_\mathbf{q} f_{-\mathbf{k}} \\ -\tilde\gamma_{\mathbf{qq}'} f_\mathbf{q} f_{-\mathbf{q}'} \end{array}\right] \end{aligned}$$

$$= 2\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\gamma_{\mathbf{q}'} f_{\mathbf{q}}(1+f_{\mathbf{q}})[\delta_{\mathbf{q},\mathbf{q}'} - M_{\mathbf{q}'\mathbf{q}}] \quad (5)$$

where the detailed balance conditions and Eq. (4) have been used in order to simplify these expressions. The factor of two comes from the contributions of in- and out-scattering processes.

Current fluctuations may be calculated using the Boltzmann-Langevin equation as follows: first we determine the local fluctuations in occupation. In the limit of long times and long length-scales, we may neglect the derivative terms on the left-hand-side of Eq. (3). The fluctuation in the distribution function is then given by

$$\delta f_{\mathbf{k}} - \delta \bar{f}_{\mathbf{k}} = \eta_{\mathbf{k}}/\gamma_{\mathbf{k}}. \quad (6)$$

This integral equation has a formal solution given by

$$\delta f_{\mathbf{k}}(\mathbf{x},t) = \int d\mathbf{q}[\mathbf{1} - \mathbf{M}]^{-1}_{\mathbf{kq}} \eta_{\mathbf{q}}(\mathbf{x},t)/\gamma_{\mathbf{q}}, \quad (7)$$

where we are using a matrix notation for functions of momentum with $[\mathbf{1}]_{\mathbf{kq}} = (2\pi)^2\delta(\mathbf{k}-\mathbf{q})$ and the inverse of a matrix $\mathbf{N}$ defined as $\int d\mathbf{q}[\mathbf{N}]^{-1}_{\mathbf{kq}}[\mathbf{N}]_{\mathbf{qk}'} = [\mathbf{1}]_{\mathbf{kk}'}$.

The resulting fluctuation in current is given by:

$$\delta\mathbf{j} = \int d\mathbf{p}\mathbf{v}_{\mathbf{p}}\delta f_{\mathbf{q}}, \quad (8)$$

where $\mathbf{v}_{\mathbf{k}} = \partial_{\mathbf{k}}\epsilon_{\mathbf{k}}$. Using the noise correlations from Eq. (5), after some algebra, we obtain the following expression for the correlation of current fluctuations in thermal equilibrium [14]:

$$\begin{aligned}\langle \delta j_\alpha(\mathbf{r},t)\delta_\beta(\mathbf{r}',t')\rangle &= \delta_{\alpha,\beta}\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\int d\mathbf{p}d\mathbf{q}\mathbf{v}_{\mathbf{p}}.\mathbf{v}_{\mathbf{q}} \\ &\times[\mathbf{1}-\mathbf{M}]^{-1}_{\mathbf{pq}} f_{\mathbf{q}}(1+f_{\mathbf{q}})/\gamma_{\mathbf{q}} \quad (9)\end{aligned}$$

This result should be compared with the result for fermions[7] where the Bose enhancement factor $f+1$ is replaced by a Fermi factor $1-f$. The low frequency current noise S_j can now be expressed as:

$$S_j = \int_{-\infty}^{\infty} dt d\mathbf{r}\langle \mathbf{j}(\mathbf{r},t).\mathbf{j}(0,0)\rangle \quad (10)$$

In Thermal equilibrium, the current fluctuations reduce to the Johnson result $S_j = 4\sigma_T T$, with the universal conductivity σ_T determined by a linear response expansion of the Boltzmann equation (2)[16]. As the electric field is increased away from zero, symmetry under $\mathbf{E} \to -\mathbf{E}$ requires that the lowest order correction to this result must be proportional to $\mathbf{E}^2$. A closed form expression for this correction may be obtained by including the lowest order, static distortion of the distribution function by the electric field, δf^s[16]. The resulting expression is cumbersome and not very revealing. We note, however, that the corrections to the distribution function are proportional to powers of $\hbar ceE/(kT)^2$ (to see this, rescale the momentum integral in the expression for δf^s in [16] and use the fact that for critical, relativistic bosons $|\mathbf{v}_{\mathbf{k}}| = c$). The current noise is, therefore, expected to be a universal scaling function of $\hbar ceE/(kT)^2$;

$$S_j = Tf\left[\hbar ceE/(kT)^2\right] = T\Phi[T_{eff}(E)/T], \quad (11)$$

where we have defined $T_{eff}(E) = \sqrt{\hbar ceE}\pi^2/4k$ (the reason for the numerical factor in this expression will become apparent later). The low-temperature limit of this scaling function is $\Phi(0) = 4\sigma_T$, where $\sigma_T = 0.2154e^2/h$ given by the universal conductance found in thermal equilibrium[1, 2]. Next, we turn to a calculation of the high-field limit of this scaling function.

Out-of-Equilibrium. Under the application of a strong electric field, the system is driven far from thermal equilibrium. In this situation, the Boltzmann equation (2) misses important field-induced tunneling processes (precisely analogous to Zener breakdown or the Schwinger mechanism). A Boltzmann equation for this situation was previously derived within a 1/N-expansion in Ref.[6];

$$[\partial_t + (e\mathbf{E}/\hbar).\partial_{\mathbf{k}}]\, f(\mathbf{k}) = -\Gamma_{\mathbf{k}} f(\mathbf{k}) + e^{-\pi\hbar c\epsilon^2_{\mathbf{k}}/eE}\delta\left(\frac{\hbar\mathbf{k}.\mathbf{E}}{eE^2}\right), \quad (12)$$

where to lowest order in 1/N[15], $\epsilon^2_{\mathbf{k}} = \mathbf{k}^2$ and

$$\begin{aligned}\Gamma_{\mathbf{k}}(t) &= \frac{8c}{N}\frac{\sqrt{|\mathbf{k}|+k_\parallel}}{\epsilon_{\mathbf{k}}}\int d\mathbf{k}'\frac{\sqrt{|\mathbf{k}'|+k'_\parallel}}{\epsilon_{\mathbf{k}'}} f(\mathbf{k}',t) \\ &= \frac{2}{\pi}\frac{1}{\sqrt{Nk_\parallel}}(eE/\hbar)^{3/4}c^{1/4}. \quad (13)\end{aligned}$$

Eq.(12) may be integrated to obtain an explicit solution for the out-of equilibrium distribution function[6, 17].

Following the same procedure as previously, we may derive a Boltzmann-Langevin equation describing fluctuations in occupation number. This has a simpler form than in thermal equilibrium since there is just one process that dominates the contribution to the fluctuations in occupation number in this strongly out-of-equilibrium limit. The result is (suppressing position and time labels)

$$\begin{aligned}[\partial_t + \mathbf{v}.\partial_{\mathbf{r}} + \mathbf{E}.\partial_{\mathbf{k}}]\,\delta f_{\mathbf{k}} &= -\Gamma_{\mathbf{k}}\delta f_{\mathbf{k}} + \eta_{\mathbf{k}} \\ \langle \eta_{\mathbf{q}}(\mathbf{r},t)\eta_{\mathbf{q}'}(\mathbf{r}',t')\rangle &= (2\pi)^2\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\delta(\mathbf{q}-\mathbf{q}')\Gamma_{\mathbf{q}} f_{\mathbf{q}} \quad (14)\end{aligned}$$

Solving, as before, in the limit of long times and large distances, the fluctuation in occupation number is given by $\delta f_{\mathbf{q}} = \eta_{\mathbf{q}}/\Gamma_{\mathbf{q}}$ and the correlation of fluctuations in current are given by

$$\begin{aligned}&\langle j_\alpha(\mathbf{r},t)j_\beta(\mathbf{r}',t')\rangle \\ &= 2e^2\delta_{\alpha,\beta}\int d\mathbf{p}d\mathbf{q}\frac{\mathbf{v}_{\mathbf{p}}.\mathbf{v}_{\mathbf{q}}}{\Gamma_{\mathbf{p}}\Gamma_{\mathbf{q}}}\langle \eta_{\mathbf{p}}(\mathbf{r},t)\eta_{\mathbf{q}}(\mathbf{r}',t')\rangle \\ &= 2e^2c^2\delta_{\alpha,\beta}\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\int d\mathbf{p}\frac{f_{\mathbf{p}}}{\Gamma_{\mathbf{p}}}, \quad (15)\end{aligned}$$

where we have used the fact that $\mathbf{v}_{\mathbf{p}}^2 = c^2$ for relativistic bosons in the large N limit and the factor of two comes from the contribution of particles and holes. Using the explicit form of the non-equilibrium distribution function and performing the integrals, one obtains the high-field contribution to current noise as

$$\langle j_\alpha(\mathbf{r},t) j_\beta(\mathbf{r}',t')\rangle \;=\; \delta_{\alpha,\beta}\delta_{t,t'}\delta_{\mathbf{r},\mathbf{r}'}\left(\frac{N\pi}{8}\right)^2 e^2\sqrt{\frac{ecE}{\hbar}} \quad (16)$$

The large field current noise (16) is dramatically reduced compared with the linear **E**-field dependence of shot noise expected for uncorrelated charge carriers. It is tempting to ascribe this to an effective charge of carriers near to the quantum critical point proportional to $1/\sqrt{E}$. This effective charge shows a dramatic divergence as one approaches the quantum critical point reflecting the strong correlations of the system. Whether this picture can be carried through requires an analysis of higher order statistics of the current noise[11]. An alternative, and perhaps more revealing, interpretation of Eq. (16) is as a non-equilibrium equivalent of Johnson noise. The non-equilibrium analogue of the fluctuation dissipation relation is recovered if we identify an effective temperature $T_{eff} = \sqrt{\hbar c e E}\pi^2/4k$. This is deduced from $4\sigma_E T_{eff} = 2(N\pi/8)^2\sqrt{ecE/\hbar}$, where $\sigma_E = (N\pi/8)e^2/h$ is the large-field conductivity[6]. The current-noise scaling function has high-field asymptote $\Phi[x\to\infty] = 4\sigma_E x$. To estimate the magnitude of this effect in a physical setting such as the SI transition in MoGe thin films [18], we can take $E = 0.5$V/m, and estimate $c \sim v_F = 10^6$m/s. We obtain $T_{eff} \sim 400$mK, which is an order of magnitude larger than experimentally accessible temperatures. The current noise will, therefore, be enhanced over its thermal value.

In conclusion, we have considered the current noise at the 2-dimensional, z=1 superconductor to insulator quantum critical point. We find that this noise follows a universal scaling function, $S_j = T\Phi[T_{eff}(E)/T]$, with $T_{eff} = \sqrt{\hbar c e E}\pi^2/4$. This scaling function recovers Johnson noise in thermal equilibrium and crosses over to an unusual $\sqrt{E}$-dependent non-linear shot noise or non-equilibrium Johnson noise at high fields. This is a particular case of a more general $E^{z/(1+z)}$ scaling expected for high-field current noise. In this way, current-noise may reveal the universal non-linear scaling exponents predicted near to quantum phase transitions.

Acknowledgments: This work was completed with the support of the Royal Society, NSF DMR-0238760 and DMR-0213706, the LDRD program of LBNL under DOE grant DE-AC02-05CH11231 and the A. P. Sloan Foundation (A.V.). We would like to thank D. Gutman and N. Mason for discussions.

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[12] Explicit forms for these in the 1/N-expansion are given in Ref.[4];

$$\gamma_{\mathbf{k},\mathbf{q}} \;=\; \frac{2}{N}\int_0^\infty \frac{d\Omega}{\pi}\frac{\mathcal{I}m\Pi^{-1}(\mathbf{q}-\mathbf{k},\Omega)}{4\epsilon_{\mathbf{k}}\epsilon_{\mathbf{q}}} \times \left[\begin{array}{c}[1+n(\Omega)]\delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{q}}-\Omega)\\ +n(\Omega))\delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{q}}+\Omega)\end{array}\right]$$

and

$$\tilde{\gamma}_{\mathbf{k},\mathbf{q}} \;=\; \frac{2}{N}\int_0^\infty \frac{d\Omega}{\pi}\frac{\mathcal{I}m\Pi^{-1}(\mathbf{k}+\mathbf{q},\Omega)}{4\epsilon_{\mathbf{k}}\epsilon_{\mathbf{q}}} \times[1+n(\Omega)]\delta\left(\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{q}}-\Omega\right)$$

$\Pi(\mathbf{q},\Omega)$ is the polarization loop for particles or anti-particles and is determined by the critical scattering; we will give no further details of it here except to use its critical form in our final calculations. For a complete comparison with the expressions given in Ref.[4], one must make use of the identity $n(\Omega) = e^{-\Omega/T}(1+n(\Omega))$.

[13] In fact, there is a subtlty here. Particle-hole scattering induces correlations between noise in the particle and hole channels. Eq. (5) incorporates these effects such that we may treat particle and hole scattering as if they were independent obtaining the correct result. Strictly, the noise correlations are given by (suppressing space and time labels)

$$\begin{aligned}\langle \eta^a_{\mathbf{q}}\eta^a_{\mathbf{q}'}\rangle \;&=\; 2\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\left[\begin{array}{c}(2\pi)^2\delta_{\mathbf{q},\mathbf{q}'}\int d\mathbf{k}\gamma_{\mathbf{q},\mathbf{k}}f_{\mathbf{q}}(1+f_{\mathbf{k}})\\ -\gamma_{\mathbf{q},\mathbf{q}'}f_{\mathbf{q}}(1+f_{\mathbf{q}'})\\ +(2\pi)^2\delta_{\mathbf{q},\mathbf{q}'}\int d\mathbf{k}\tilde{\gamma}_{\mathbf{q},\mathbf{k}}f_{bfq}f_{-\mathbf{k}}\end{array}\right]\\ \langle \eta^a_{\mathbf{q}}\eta^b_{\mathbf{q}'}\rangle \;&=\; 2\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\left[\ \tilde{\gamma}_{\mathbf{q},\mathbf{q}'}f_{\mathbf{q}}f_{-\mathbf{q}'}\ \right]\end{aligned}$$

writing the total contribution to the current fluctuation as

$$\delta\mathbf{j} = \int d\mathbf{p}\mathbf{v}_{\mathbf{p}}\frac{\eta^a_{\mathbf{p}}-\eta^b_{\mathbf{p}}}{\gamma_{\mathbf{p}}}$$

with these noise correlators we recover the results derived in the text.

[14] In matrix notation the equation (5) reads:

$$\langle \eta_{\mathbf{q}}(\mathbf{r},t)\eta_{\mathbf{q}'}(\mathbf{r}',t')\rangle = 2\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\gamma_{\mathbf{q}'}f_{\mathbf{q}}(1+f_{\mathbf{q}})[\mathbf{1}-\mathbf{M}]_{\mathbf{q}'\mathbf{q}}$$

Combining this with Eq. (7) we find the fluctuations of the distribution function are correlated as:

$$\langle \delta f_{\mathbf{p}}(\mathbf{r},t)\delta f_{\mathbf{q}}(\mathbf{r}',t')\rangle = 2\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}[\mathbf{1}-\mathbf{M}]^{-1}_{\mathbf{pq}}\frac{f_{\mathbf{q}}(1+f_{\mathbf{q}})}{\gamma_{\mathbf{q}}}$$

hence correlators of the current fluctuations (8) are:

$$\langle \delta j_{\alpha}(\mathbf{r},t)\delta_{\beta}(\mathbf{r}',t')\rangle = \delta_{\alpha,\beta}\delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'}\int_{\mathbf{pq}}[\mathbf{1}-\mathbf{M}]^{-1}_{\mathbf{pq}}f_{\mathbf{q}}(1+f_{\mathbf{q}})/\gamma_{\mathbf{q}}$$

[15] Within the 1/N-expansion for the out-of-equilibrium system, the gap does not contribute to the current response at lowest order in $1/N$ and is neglected here (See also [6]).

[16] The linear response conductivity may be calculated from the Boltzmann equation (2) in the usual way. Expanding to linear order in the applied electric field, we find

$$\delta f^s_{\mathbf{k}} - \int d\mathbf{q}M_{\mathbf{kq}}\delta f^s_{\mathbf{q}} = \frac{1}{\gamma_{\mathbf{k}}}\mathbf{E}.\partial_{\mathbf{k}}f_{\mathbf{k}},$$

where δf^s now indicates the static, linear distortion of the distribution from the thermal distribution f in response to the electric field. We may formally invert this integral with the result

$$\delta f^s_{\mathbf{k}} = \int d\mathbf{q}[\mathbf{1}-\mathbf{M}]^{-1}_{\mathbf{kq}}\frac{1}{\gamma_{\mathbf{q}}}\mathbf{E}.\partial_{\mathbf{q}}f_{\mathbf{q}}$$

Finally, using $\mathbf{j} = \int d\mathbf{k}\mathbf{v}_{\mathbf{k}}\delta f^s_{\mathbf{k}}$, and $\partial_{\mathbf{q}}f_{\mathbf{q}} = \mathbf{v}_{\mathbf{q}}(1+f_{\mathbf{q}})f_{\mathbf{q}}/kT$ we may deduce the following linear response solution for the conductivity in thermal equilibrium:

$$kT\sigma \;=\; 2\int d\mathbf{p}d\mathbf{q}\mathbf{v}_{\mathbf{p}}.\mathbf{v}_{\mathbf{q}}[\mathbf{1}-\mathbf{M}]^{-1}_{\mathbf{pq}}f_{\mathbf{q}}(1+f_{\mathbf{q}})/\gamma_{\mathbf{q}}$$

the factor of 2 arising from the contribution of both particles and anti-particles to the conductivity. This may be compared against Eq.(9). When calculated within a 1/N-expansion, the result of carrying out these integrals is a universal conductivity in thermal equilibrium, given by $\sigma_T = N0.1077e^{*2}/h$[1].

[17] The explicit solution for the distribution function is $f_{\mathbf{k}} = \theta(K_{\parallel})\exp\left[-\pi K_{\perp}^2 - \gamma\sqrt{2K_{\parallel}}\right]$, where the momenta have been rescaled according to $\mathbf{K} = \mathbf{k}/\sqrt{eE/\hbar c}$ and γ is determined by $\Gamma_{\mathbf{k}} = \gamma(eE/\hbar)^{3/4}c^{1/4}/\sqrt{2k_{\parallel}}$. In terms of these variables, Eq. (12) may be integrated to obtain the above solution. The integrals in Eq. (13) may be carried out noting that $k_{\parallel} \gg k_{\perp}$ to obtain $\gamma^2 = 8/N\pi^2$.

[18] N. Mason and A. Kapitulnik, Phs. Rev. Lett. **82**, 5341 (1999).